Methods for Efficient Computation of Neutron Multiplicity Distributions

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Abstract

Neutron fingerprinting by neutron multiplicity measurement is a proven technique to establish unique signatures for identification, accountability and control of SNM in nuclear safeguards applications. Computational modeling of neutron multiplicity distributions is a vital complement to experiments but has traditionally relied on either grossly simplified models (singles, double, and triples) or computationally demanding Monte Carlo simulation. The ability to accurately and efficiently compute multiplicity distributions is recognized to be an essential component of the nuclear safeguards toolkit. We present a hierarchical point-kinetic methodology based on a backward Master equation formulation to numerically compute accurate time-gated neutron count number probability distributions of arbitrary order and statistical moments of these distributions. For low count numbers, we show that the multiplicity distribution can be efficiently computed by direct numerical solution of the sequential count probability equations generated from the backward Master equation. For high count numbers, on the other hand, a generalized Laguerre polynomial representation with a gamma distribution weight function is shown to accurately reconstruct the multiplicity distribution, requiring only low order statistical moments as input. For intermediate count numbers, a maximum entropy reconstruction is found to reproduce the count distribution very accurately when both moments and low-order discrete count probabilities are incorporated as constraints. Numerical results are presented for a number of different scenarios and benchmarked for accuracy against a stochastic simulation algorithm (SSA), a system state-updating Monte Carlo method that is more efficient than single-event Monte Carlo in point geometries.

Introduction

The ability to quickly and accurately characterize unknown material samples is of high importance to nuclear nonproliferation and international safeguards. Non-destructive assay (NDA) techniques, such as coincidence counting and Neutron Multiplicity Counting (NMC), are widely used to extract quantitative and qualitative information from neutrons emitted by unknown samples of interest. For instance, in NMC [1], neutron multiplicity distributions provide unique nuclear signatures for material property identification where an extended tail in the count number probability distribution is observed that is a characteristic of and sensitive to the amount of SNM present. Moreover, accurate and efficient methods for computing neutron multiplicity distributions are complementary to experimental measurements and help understand the fundamental interactions and transport processes underlying nuclear signatures. Theoretical modeling approaches that are intermediate between gross-system level models [2] and high-fidelity but computationally demanding Monte Carlo simulations [3] are of particular interest in this regard. While low order statistical moments of the multiplicity distribution are relatively inexpensive to compute, and in conjunction with crude models yield information on sample mass and composition [2], the multiplicity distribution itself provides a more detailed statistical characterization. However, multiplicity distributions can be efficiently computed only for relatively small count numbers ($\sim$ tens), with the computational effort growing nonlinearly with increasing count numbers.

This work provides an overview of an approach for accurately computing multiplicity distributions that exploits unique features of the distribution. The computational model consists of a dynamic backward Master equation for a time-gated count number probability distribution and its statistical moments [5].

Based on the max number of counts, the count distribution is computed by direct numerical solution of this backward Master equation or reconstructed from statistical moments and low-order discrete count probabilities. Two reconstruction methods are presented: a generalized Laguerre polynomial representation, and a maximum entropy method. Numerical results are presented for a number of different scenarios and benchmarked against an implementation of the Stochastic Simulation Algorithm (SSA) [4], a system state-updating Monte Carlo method that is more efficient than single-event Monte Carlo in lumped geometries.

SSA Monte Carlo

The Stochastic Simulation Algorithm (SSA)[6] is used to simulate the stochastic state of a point kinetic system and obtain the number of neutron counts within a detector time gate. The stochastic state of the simulation system is represented by the instantaneous number of neutrons (n) within the assembly and the number of neutron counts (m) registered by an external nearby detector during a fixed time period, although the state may be readily expanded to accommodate other stochastic properties of interest, e.g., number of photons. The state is evolved over time in response to different types of neutron collisions with characteristic reaction probability $\lambda_\mu\, dt$ in a short time dt, where $\mu = f$ (fission), c (capture), and l (leakage). Let $P_{n,m}(t)$, $n \geq 0$, be the probability that n neutrons exist at time t with m neutrons being detected up to that time (or in a time interval), $S\Delta t$ the probability of a spontaneous fission event in a short time Δt with multiplicity distribution q_ν, $\nu = 0, 1, 2, \ldots \widehat{\nu}_s$, and p_ν, $\nu = 0, 1, 2, \ldots \widehat{\nu}$ the induced fission neutron multiplicity distribution. Given a Markovian system, he SSA algorithm may be constructed from the following probability balance over all independent and mutually exclusive events:

$$\begin{aligned} P_{n,m}(t+\Delta t) &= (1-(S+n\lambda_T)\Delta t)P_{n,m}(t) + \lambda_c \Delta t(n+1)P_{n+1,m}(t) \\ &+ \lambda_l \Delta t(n+1)P_{n+1,m-1}(t) + S\Delta t \sum_{\nu=0}^{\widehat{\nu}_S} \Big[q_\nu P_{n-\nu,m}(t) \Big] \\ &+ \lambda_f \Delta t \sum_{\nu=0}^{\widehat{\nu}} \Big[p_\nu (n+1-\nu) P_{n+1-\nu,m}(t) \Big], \end{aligned} \tag{1}$$

given some initial distribution at $t = 0$.The first term on the right hand side of Eq.(1) expresses the probability that there is no interaction in Δt and the neutron does not leak from the system. The remaining probabilities and event outcomes are described in Table 1.

Table 1: State change probabilities and outcomes

Term #	Event	Probability/Time	State Change
2	Source	$S\, q_\nu$	$n \to n+\nu$ $m \to m$
3	Capture	$n\, \lambda_c$	$n \to n-1$ $m \to m$
4	Leakage	$n\, \lambda_l$	$n \to n-1$ $m \to m+1$
5	Fission	$n\, \lambda_f\, p_\nu$	$n \to n-1+\nu$ $m \to m$

The simulation proceeds by sampling a time to the next state updating event, determining a particular interaction according to the probabilities in Table 1, and updating the state accordingly. The constructed

algorithm was used to reproduce previously published results [2] as illustrated in Fig.1. Simulating a single source event from a sample of composition 80 wt% Pu239 and 20 wt% Pu240 for sample masses of 335g, 2680g, and 9047g, the SSA accurately reproduced the resulting neutron count distribution. The point kinetic data used for the simulation was obtained from a standard adjoint weighted 1D spherical geometry transport calculations. Considering that the isotope composition did not change across all three samples, the rates at which fission and capture occurred in the sample were constant and equated to $5.16 \times 10^7 s^{-1}$ and $1.51 \times 10^6 s^{-1}$, respectively. The remainder of the point kinetic data, such as source and leakage rates, are given in Table 2 below.

Table 2: Point kinetic data

Mass	Source Rate	Leakage Rate
335g	7.01×10^4	5.39×10^8
2680g	5.61×10^5	2.10×10^8
9047g	1.89×10^6	1.11×10^8

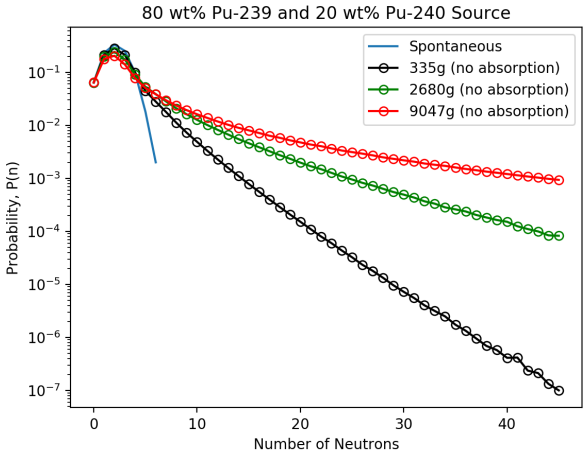


Figure 1: Neutron count distribution

Backward Master Equation

As previously mentioned, the backward Master equation is more appropriate for describing the count distribution directly. Let $P_n(D|s)$ be the probability that n counts are registered in a finite time interval $D : \{t_{min}, t_{max}\}$ (e.g., detector time gate) conditional on one neutron existing in the sample at an earlier time s, i.e., the single chain count distribution. Similarly, let $\Theta_n(D|s)$ be the probability that n counts are registered conditional on an intrinsic random source of strength S being first turned on at an earlier time s. Then, it is readily shown using first-step probability balance arguments [7, 8] that both distributions satisfy backward Master equations in the form

$$-\frac{\partial}{\partial s}P_n(D|s) = -\lambda_T P_n(D|s) + \lambda_c \delta_{n,0} + \lambda_l[\delta_{n,1} I_D(s) + \delta_{n,0} I_{\overline{D}}(s)] + \lambda_f \sum_{\nu=0}^{\widehat{\nu}} p_\nu \sum_{|\vec{n}_\nu|=n} \prod_{j=1}^{\nu} P_{n_j}(D|s), \tag{2}$$

$$-\frac{\partial}{\partial s}\Theta_n(D|s) = -S\Theta_n(D|s) + S \sum_{\nu=0}^{\widehat{\nu}_S} q_\nu \sum_{|\vec{n}_\nu|+|\vec{m}_\nu|=n} \prod_{j=1}^{\nu} P_{n_j}(D|s), \Theta_{m_j}(D|s), \tag{3}$$

with final conditions $P_n(D|t_{\max}) = \delta_{n,0}$ and $\Theta_n(D|t_{\max}) = \delta_{n,0}$, and defining $|\vec{n}_\nu| = n_1 + n_2 + \ldots + n_\nu$ and similarly $|\vec{m}_\nu|$. In Eqs. (2) and (3), $I_D(s)$ is the indicator function, defined to be unity for $s \in D$ and zero otherwise, and $I_{\overline{D}}(s)$ its complement. This closed set of differential equations can be solved recursively for the count probability distribution and statistical moments increasing count numbers. To this end, it is convenient to express Eqs (2) and (3) in terms of the probability generating functions defined by the discrete transformation:

$$G(z; D|s) = \sum_{n=0}^{\infty} z^n P_n(D|s), \quad H(z; D|s) = \sum_{n=0}^{\infty} z^n \Theta_n(D|s), \tag{4}$$

where z is the transform variable restricted to the range $0 \leq z \leq 1$ to ensure convergence of the summations. Applying this discrete transform to Eqs.(2) and (3) gives

$$-\frac{\partial}{\partial s}G(z;D|s) = -\lambda_T G(z;D|s) + \lambda_c + \lambda_l[zI_D(s) + I_{\overline{D}}(s)] + \lambda_f \sum_{\nu=0}^{\widehat{\nu}} p_\nu G^\nu(z;D|s), \tag{5}$$

$$-\frac{\partial}{\partial s}H(z;D|s) = -SH(z;D|s) + S\Big[\sum_{\nu=0}^{\widehat{\nu}_S} q_\nu G^\nu(z;D|s)H(z;D|s)\Big], \tag{6}$$

with final conditions $G(z;D|t) = 1$ and $H_n(z;D|t) = 1$.

Count Probability Equations

Equations for discrete count probabilities are readily obtained from the BME noting the relationships

$$P_k = \frac{1}{k!}\frac{\partial^k}{\partial z^k}G(z;D|s)\Big|_{z=0}, \qquad \Theta_k = \frac{1}{k!}\frac{\partial^k}{\partial z^k}H(z;D|s)\Big|_{z=0}, \tag{7}$$

where $G(z;D|s)$ and $H(z;D|s)$ are the generating function equations for $P_n(D|s)$ and $\Theta_n(D|s)$, respectively. Differentiating Eqs. (5) and (6) to the appropriate order yields the count probability equations:

$$-\frac{\partial}{\partial s}P_k(D|s) = -\lambda_T P_k + \frac{1}{k!}\Big[\lambda_l I_D(s)\delta_{k,1} + \big(\lambda_c + \lambda_l I_{\bar{D}}(s)\big)\delta_{k,0} + \lambda_f \sum_{\nu=0}^{\widehat{\nu}} p_\nu I_k^\nu\Big] \tag{8}$$

$$-\frac{\partial}{\partial s}\Theta_k(D|s) = -S\Theta_k + \frac{1}{k!}S\sum_{\nu=0}^{\widehat{\nu}_S} p_\nu J_k^\nu. \tag{9}$$

where the derivative terms associated with the branching process have been represented as

$$I_k^\nu = \frac{\partial^k}{\partial z^k}G^\nu(z;D|s)\Big|_{z=0}, \qquad J_k^\nu = \frac{\partial^k}{\partial z^k}\big[G^\nu(z;D|s)H(z;D|s)\big]\Big|_{z=0} \tag{10}$$

To proceed further, it becomes necessary to derive some representation of the branching terms I_k^ν and J_k^ν. Noting patterns that become evident from lower order terms, the following recursive relationship can be derived

$$I_k^\nu = \begin{cases} P_0^\nu, & k = 0 \\ \nu\displaystyle\sum_{j=0}^{k-1}\binom{k-1}{j} I_j^{\nu-1}(k-j)!P_{k-j}, & k \geq 1 \end{cases} \tag{11}$$

$$J_k^\nu = \sum_{j=0}^{k}\binom{k}{j} I_j^\nu (k-j)!\Theta_{k-j}, \qquad k \geq 0 \tag{12}$$

It is noted that the equation for P_0, the probability of observing no counts, is standalone but nonlinear while the equations for P_n, $n \geq 1$, are linear but fully coupled. Nevertheless, both terms, I_k^ν and J_k^ν, were numerically evaluated by writing a recursive lambda function in C++. In principle, this algorithm enables the branching terms in the count probability equations to be obtained explicitly up to an arbitrary order. In practice, the number of such terms grows explosively as the count order k increases and the resulting expression-swell creates a numerical challenge when k is larger than a few tens. The count probability equations were solved sequentially using an adaptive Runge-Kutta scheme and backwards integration from the later edge of the time gate to an initial time. For illustration, a sample of composition 80 wt% ^{239}Pu and 20 wt% ^{240}Pu with a mass of 335g is considered. Fig. 2 compares the distribution generated over a

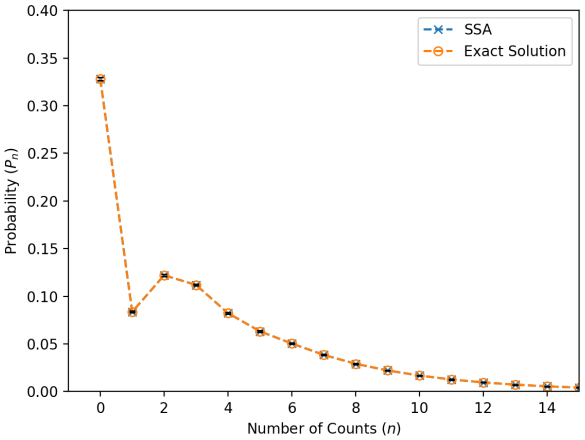


Figure 2: BME exact neutron count distribution compared against SSA Monte Carlo

Table 3: BME Direct Solve Computation Times

n	CPU Times (s)
0	0.015
5	0.46
10	10.1
11	19.3
12	33.4
13	60.3
14	96.2
15	**157.6**
SSA	**36**

$17\mu s$ time gate computed by solving Eqs. (8) and (9) numerically against an SSA Monte Carlo simulation. The relative standard deviation is less than 1% for the low count numbers (3 - 4 counts) and increases to a maximum of 5% in the tail of the distribution (14 - 15 counts). The time required to solve each set of equations is also represented in Table 3 illustrating the rate at which the computation time scales with increasing k. When the number of counts is large, it is clear that such methods are less efficient than monte carlo approaches. For such scenarios, methods based on a knowledge of low order statistical moments, which are easily solved, prove more efficient if not as accurate, as is shown next.

Reconstruction Methods

Despite the direct numerical solution of Eqs. (2) and (3) being quite efficient for low count numbers, solving for the individual count number probability equations becomes tedious for large count numbers. This trend is also observed when solving for the statistical moments of the distribution which, by definition, contain information about the entire distribution. Considering that more information is contained in the first few moments as opposed to the first few counts of the probability distribution, it is natural to inquire if the distribution can be reconstructed from a finite, preferably small, number of moments. Statistical moments of the distribution function, from which gross sample properties can be extracted [2], are readily obtained from the BME and easily computed. Multiplicity moments M_k of the count distribution may be obtained directly from the generating function by taking derivatives of Eq. 4, giving

$$M_k(D|s) = \overline{n(n-1)\dots(n-k+1)} = \frac{1}{k!}\frac{\partial G(z:D|s)}{\partial z^k}\Big|_{z=1}. \tag{13}$$

Similarly, multiplicity moments when a source is present are given by

$$M_k^S(D|s) = \overline{n(n-1)\dots(n-k+1)} = \frac{1}{k!}\frac{\partial H(z:D|s)}{\partial z^k}\Big|_{z=1} \tag{14}$$

Specifically, when $k=1$ the equations above give the mean number of counts while $k=2$ gives the second factorial moment $\overline{n(n-1)}$ from which the variance can be calculated. Taking the k^{th}-order derivative of Eqs.5 and 6 gives the moment equations

$$-\frac{\partial}{\partial s}M_k(t|s) = -\lambda_T M_k(t|s) + \lambda_l I_D(s)\delta_{k,1} + \lambda_f \sum_{\nu=0}^{\widehat{\nu}} p_\nu \frac{\partial^k}{\partial z^k}G^\nu(z;t|s)\Big|_{z=1} \tag{15}$$

$$-\frac{\partial}{\partial s}M_k^S(t|s) = -SM_k^S(t|s) + S\sum_{\nu=0}^{\widehat{\nu}_S} q_\nu \frac{\partial^k}{\partial z^k}\Big[G^\nu(z;t|s)H(z;t|s)\Big]\Big|_{z=1} \tag{16}$$

Similar to the terms associated with the branching process in Eqs. (8) and (9), the inhomogeneous or source terms depend only on the previous moments. This facilitated the ability to solve these equations sequentially by backward integration in time to obtain the count multiplicity moments of desired order. From that, two methods were considered for the reconstruction of probability distributions which are discussed within this section.

Maximum Entropy Method

The first method is the maximum entropy method. A widely used technique for reconstructing a probability distribution from a knowledge of a finite number of moments of the distribution [9]. The information or Shannon entropy is defined by the discrete functional

$$S[P_n] = -\sum_{n=0}^{\infty} P_n \ln(P_n), \tag{17}$$

which when maximized after incorporating the moments as constraints using Lagrange multipliers provides the least biased distribution for the given information. The procedure yields a nonlinear system of equations for the Lagrange multipliers which can be solved using an iterative method such as Newton [10]. Incorporating the first K moments of the true distribution as constraints, the following functional is obtained for the reconstructed distribution $\tilde{P}_n$

$$\tilde{P}_n = S[\tilde{P}_n] - \sum_{k=0}^{K}\left[\lambda_k\Big(\sum_{n=0}^{\infty}\Big(\frac{n}{\bar{n}}-1\Big)^k\tilde{P}_n - \mu_k\Big)\right]; \qquad \mu_k = \sum_{n=0}^{\infty}\Big(\frac{n}{\bar{n}}-1\Big)^k P_n \tag{18}$$

where μ_k are the normalized central moments of the true distribution P_n and λ_k are the Lagrange multipliers. It is noted that in our application we found that using normalized central moments as opposed to raw moments significantly improved the numerical convergence of the Newton iteration. The functional above, when maximized, yields the equation for the approximated distribution

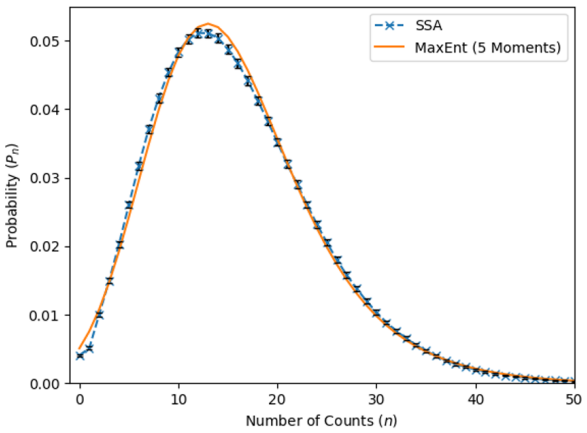
$$\tilde{P}_n = \exp\Big(-1 + \sum_{k=1}^{K} -\lambda_k\Big(\frac{n}{\bar{n}}-1\Big)^k\Big) \tag{19}$$

Figs. 3a and 3b illustrate the reconstruction of the distribution using this method for sample masses of 335g and 2680g at 84 μs (negligible zero-count probability) and 6 μs (significant zero-count probability) time gates, respectively. The SSA relative standard deviation ranges from less than 2% around the peak of both distributions to 15% in the tails. The reconstruction appears to accurately reproduce the probability distribution for cases when the number of counts is intermediate and the probability of zero counts is negligible (Fig. 3a). When the probability of zero counts is significant, the maximum entropy method captures the tail of the distribution, but fails to reproduce the observed nonmonotonic structure of the distribution (Fig. 3b). However, the nonmonotonicity occurs at the low counts which can be easily obtained as shown in the previous section. This gave rise to the idea of incorporating the true low order count probabilities as discrete constraints. In order to additionally incorporate the n^* discrete constraints, we express

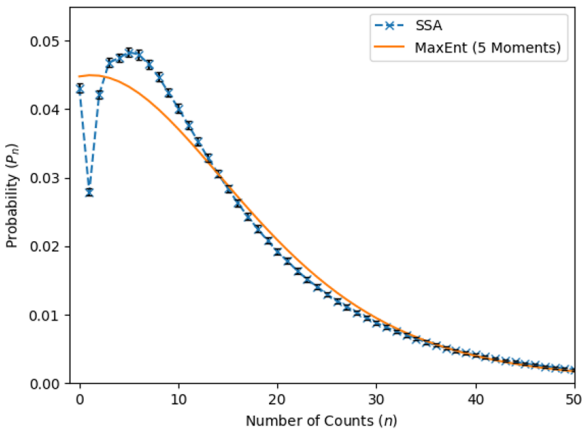
$$\sum_{n=0}^{\infty}\Big(\frac{n}{\bar{n}}-1\Big)^k\tilde{P}_n - \mu_k = \sum_{n=0}^{n^\star}\Big(\frac{n}{\bar{n}}-1\Big)^k P_n + \sum_{n=n^\star+1}^{\infty}\Big(\frac{n}{\bar{n}}-1\Big)^k\tilde{P}_n - \mu_k$$

where P_n are the exact count probabilities obtained by solving Eqs.(8) and (9) up to order $n^\star$. Furthermore, we define the modified constraints $\alpha_k^{(n^\star)}$ as

$$\alpha_k^{(n^\star)} = \mu_k - \sum_{n=0}^{n^\star}\Big(\frac{n}{\bar{n}}-1\Big)^k P_n \tag{20}$$



(a) 335g sample source



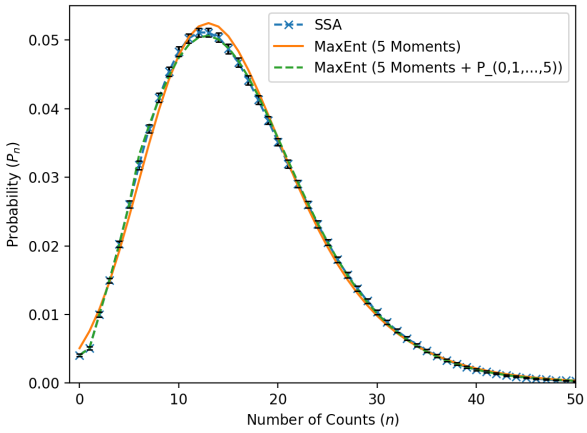
(b) 2680g sample source

Figure 3: Maximum entropy reconstructed neutron count distribution

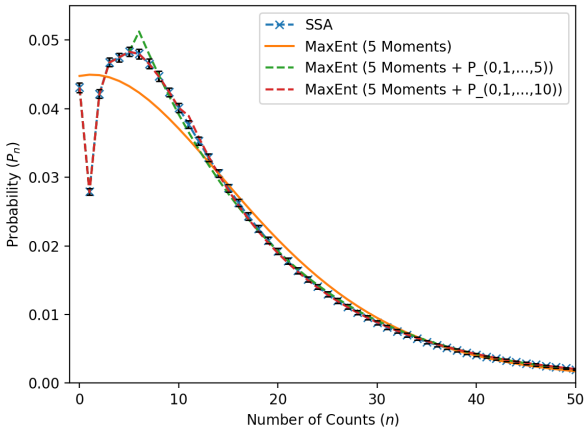
which, when inserted in the modified functional and the latter maximized, finally gives the approximated distribution:

$$\tilde{P}_n = \begin{cases} P_n, & 0 \leq n \leq n^\star \\ \exp\left(-1 + \sum_{k=1}^{K} -\lambda_k \left(\frac{n}{\bar{n}} - 1 \right)^k \right), & n > n^\star \end{cases} \tag{21}$$

For the case illustrated in Fig. 3a, although preserving five moments yields sufficient accuracy, the reconstruction accuracy is significantly enhanced when the distribution is additionally constrained with as little as the first six discrete count probabilities, $P_0 - P_5$. As for the case when the probability of observing zero counts is significant, incorporating the true low order count probabilities as discrete constraints had a major impact. Significant improvement was obtained with just five discrete count probabilities, but the distribution reconstructed with ten is practically indistinguishable from the SSA distribution. Figs. 4a and 4b illustrate the reconstruction of the distribution in Figs. 3a and 3b accounting for both moments and discrete count probabilities as constraints. As for the difference in computation time between the maximum



(a) 335g sample source



(b) 2680g sample source

Figure 4: Maximum entropy reconstructed neutron count distribution

entropy reconstruction method and SSA MC, the moment only reconstruction illustrated in Fig. 4b took

0.02s compared to the 151s of SSA. Incorporating the first six, $P_0 - P_5$, and first eleven, $P_0 - P_11$, discrete count probabilities took 0.11s and 1.87s, respectively.

Orthogonal Polynomial Expansion Method

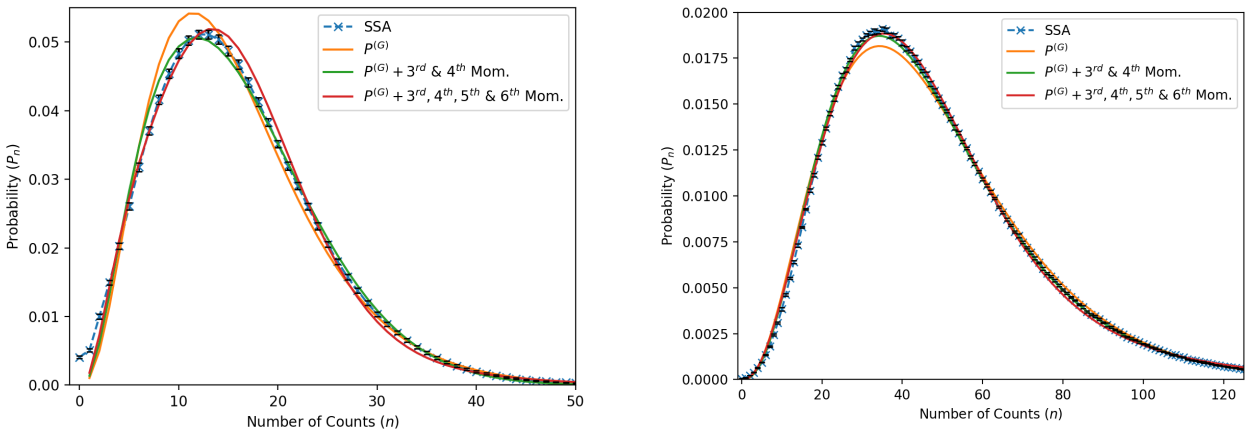
The second method considered was an Orthogonal Polynomial Expansion method. Orthogonal polynomial representations, whose associated weight functions are representative of the desired probability distribution, can yield highly accurate reconstructions of the distribution by enforcing preservation of a finite number of moments. For instance, knowledge of the first two moments (mean and variance) is sufficient to characterize a Gaussian distribution but this is only accurate when the mean number of counts is large and the variance small, a regime that is outside the domain of typical safeguards applications. However, the two-parameter gamma distribution was has been previously demonstrated to be effective in describing neutron number distributions from strongly stochastic to strongly deterministic regimes [11, 12] when the two parameters are chosen to preserve the exact mean and variance of the distribution. When applied to the count numbers, this yields:

$$P^{(G)}(n;D|s) = \left(\frac{\eta}{\bar{n}}\right)^{\eta} \frac{n^{\eta-1}}{\Gamma(\eta)} \exp\left(-\frac{n\eta}{\bar{n}}\right); \quad \eta(s) = \frac{\bar{n}^2(s)}{\sigma^2(s)}, \tag{22}$$

which is a good approximation when large count numbers dominate the distribution. Improved accuracy more broadly is achieved by expressing the count distribution as an expansion in terms of generalized Laguerre polynomials $L_k^r(x)$ which are orthogonal with a gamma distribution weight function. Thus, writing

$$P(n;D|s) \approx P^{(G)}(n;D|s)\left\{1 + \sum_{k=3}^{K} a_k L_k^{\eta-1}\left(\frac{\eta n}{\bar{n}}\right)\right\}, \tag{23}$$

the expansion coefficients a_k are obtained by requiring the distribution to yield exact moments up to order K. Note, the first two moments are automatically preserved by the gamma distribution. The distributions reconstructed from the gamma and Laguerre expansion distributions are shown in Figs.(5a) and (5b) for sample masses of 335g and 2680g of the same Pu composition with 84 μs and 19 μs time gates, respectively. The results show that a generalized Laguerre distribution is capable of producing an accurate reconstruction of the count distribution, with fewer moments required as the number of counts increases. The SSA standard deviation ranges from less than 2% for low count numbers about the peak of the distribution to 15% for the low probability high count events at the tail. The reconstruction

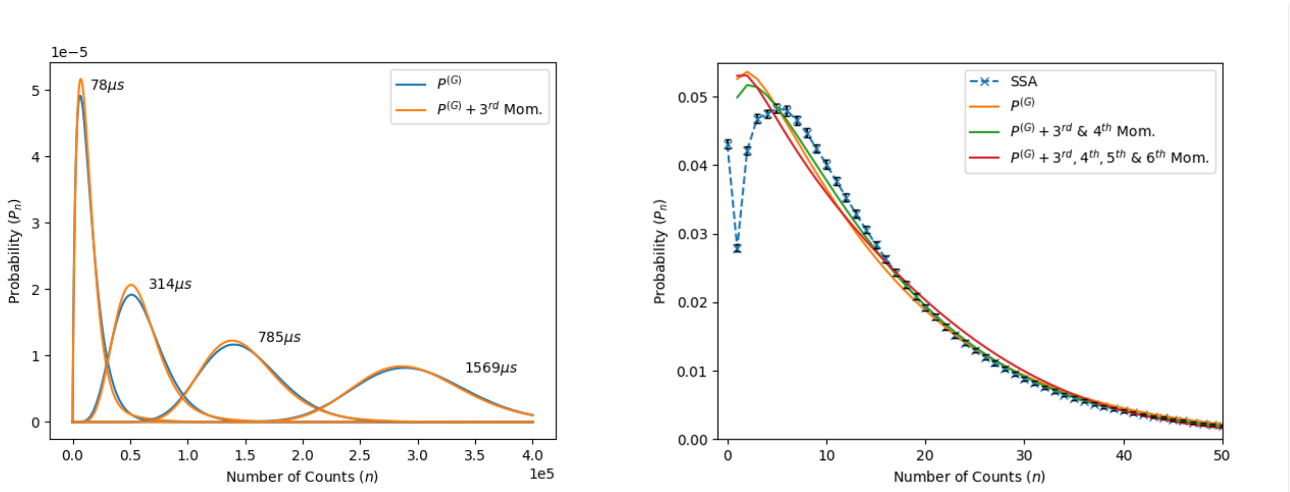


(a) 335g sample source

(b) 2680g sample source

Figure 5: Reconstructed neutron count distribution using a generalized Laguerre expansion

method mentioned here also showed drastic improvement in computation time as compared to SSA MC. For example, the time required to produce the probability distribution in Fig. 5b took a total of 456s using the SSA while the reconstruction using the orthogonal polynomial expansion took a mere 0.002s. Moreover, the efficacy of the reconstruction becomes apparent for larger time gates when the variance-to-mean ratio of counts is not large ($\eta > 1$). This is illustrated in Fig. 6a for a 9047g sample of the same Pu composition at four, increasing time gates. Under these circumstances, the gamma distribution alone accurately captures the count distribution. However, for increasing variance-to-mean such that $\eta < 1$, the underlying gamma distribution becomes monotonically decreasing and hence is not representative for very small count numbers. This is true in particular when P_0, the probability of zero counts, is not small. For such scenarios, methods such as the maximum entropy reconstruction proved to be more efficient.



(a) 9047g sample source

(b) 2680g sample source

Figure 6: Reconstructed neutron count distribution using a generalized Laguerre expansion

The results show that the Gamma pdf captures the tail of the count distributions very well but additionally preserving the third and fourth moments improves the accuracy closer to the mode. For $\eta < 1$ the Gamma distribution is monotonically decreasing and hence is not representative for very small count numbers. The results suggest that when the mean and variance yield $\eta < 1$ the pdf should be computed exactly from Eqs. (2) and (3) for low count numbers while the four moment Laguerre-reconstructed pdf can adequately represent the tails. For $\eta > 1$, on the other hand, the Laguerre-reconstructed pdf captures the entire pdf with accuracy increasing with increasing number of moments preserved.

Conclusions

Implementation of the Stochastic Simulation Algorithm was demonstrated to be a versatile, memory and performance efficient alternative to single-event Monte Carlo method for neutron multiplicity simulations in point kinetic geometry. In addition, a methodology for direct computation and moment-based reconstruction of SNM neutron multiplicity distributions has been demonstrated where the SSA was used to benchmark count probability distributions produced. Extensive numerical testing showed that direct numerical solution of the backward Master equation is practical and efficient for computing the multiplicity distribution when the count numbers are small (∼tens); for intermediate count numbers, a maximum entropy method constrained by independently and inexpensively computed low order moments and low order discrete count probabilities is also efficient and yields highly accurate reconstructions; for large count numbers, a truncated generalized Laguerre distribution constrained to produce a fixed number of exact moments also yields highly accurate representations of the multiplicity distribution, with the gamma distribution itself sufficing for very large counts. Although demonstrated here for a lumped model, the

multiplicity reconstruction approach is immediately extensible to the unlumped case where neutron phase information (position, direction, energy) is retained and where computational advantages over Monte Carlo simulation are expected to be greater still.

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